EQUATIONS OF OSCILLATIONS OF RODS OF VARIABLE COMPOSITION

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We obtain differential equations for the general case of longitudinal, torsional, and transverse oscillations of rods to some parts of which masses are being added or detached. We solve certain special problems concerning the oscillations of such rods of variable composition. In deriving generalized equations of oscillations of rods of variable composition we employ the assumption of planar sections, the assumption of small deformations, and other customary simplifications. We also employ the simplifying assumption of close action; i.e., we assume that the masses being detached and added interact with the rod only at the instant of direct contact. Forces of internal nonelastic resistance are not taken into account. We assume also that in the undeformed state the elastic axis is rectilinear and that the centers of gravity of cross sections are not displaced from their initial positions relative to the cross sections. There may be a change of mass per unit length of the rod both on account of a change in density as well as on account of a change in area of a cross section, the latter being understood to be the union of the initial area of the cross section and the areas of the parts being added and detached. In addition, with the rod there may be associated particles of variable mass distributed continuously or discretely along the length of the rod. We assume that these particles do not interact among themselves but only with the rod.

1. Equations of Longitudinal and Torsional Oscillations of Rods of Variable Composition. We assume that in the case of longitudinal oscillations the particles being added (or detached) move along the axis of the rod with speeds of the same magnitude for one and the same rod cross section. In the case of torsional oscillations the principal reaction force vectors of particles being detached or added to a section are equal to zero, and their moments are parallel to the axis of the rod. It suffices to obtain the equation of longitudinal oscillations because the equation of torsional oscillations is obtainable from the former on the basis of known analogies.

We introduce at first the equation of free oscillations of a rod of variable composition with noninteracting particles of varying mass distributed along it. Let $u(x, t)$ be the displacement of a cross section of the rod; $F(x, t)$, the cross section area; $E(x, t)$, the modulus of elasticity; $m(x, t)$, mass per unit length of the rod at time t; $m_0(x, t)$, the mass of the continuously distributed noninteracting particles associated with a unit length of the rod at time t; $v^+(x, t)$ and $v^-(x, t)$, speeds of particles of the rod being added and detached; and $v_0^+(x, t)$ and $v_0^-(x, t)$, analogous speeds for particles associated with the rod.

The equation of the longitudinal oscillations is obtained from the necessary condition for an extremum of the functional [1]

$$S_1 = \frac{1}{2}\int_{t_1}^{t_2}\int_0^l \left[(m+m_0)\left(\frac{\partial u}{\partial t}\right)^2 - EF\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial m^+}{\partial t}v^+ + \frac{\partial m^-}{\partial t}v^- + \frac{\partial m_0^+}{\partial t}v^+_0 + \frac{\partial m_0^-}{\partial t}v_0^-\right)u\right]dx\,dt$$

and has the form

$$\frac{\partial}{\partial t}\left[(m+m_0)\frac{\partial u}{\partial t}\right] - \frac{\partial}{\partial x}\left(EF\frac{\partial u}{\partial x}\right) = \frac{\partial m^+}{\partial t}v^+ + \frac{\partial m^-}{\partial t}v^- + \frac{\partial m_0^+}{\partial t}v_0^+ + \frac{\partial m_0^-}{\partial t}v_0^- \tag{1.1}$$

Volgograd. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 1, pp. 103-108, January-February, 1972. Original article submitted March 9, 1971.

where l and x are, respectively, the length of the rod and a coordinate measured from an arbitrary one of its ends.

The mass $m(x, t)$ of the rod per unit length and the mass $m_0(x, t)$ of the continuously distributed noninteracting particles associated with a unit length of the rod at time t are given, respectively, by

$$m(x,t) = m^\circ(x) + m^+(x,t) + m^-(x,t)$$
$$m_0(x,t) = m_0^\circ(x) + m_0^+(x,t) + m_0^-(x,t) \tag{1.2}$$

Here $m^+(x, t) \geq 0$, $m^-(x, t) \leq 0$ are the masses of the particles being added and detached per unit length of the rod at the time t; $m_0^+(x, t) \geq 0$, $m_0^-(x, t) \leq 0$ are the masses of noninteracting particles being added and detached per unit length of the rod at the time t; $m^\circ(x)$ is the initial mass of a unit length of the rod; $m_0^\circ(x)$ is the initial mass of mutually noninteracting particles associated with a unit length of the rod.

At the initial instant $t = 0$

$$m^+(x,0) = m^-(x,0) = 0, \qquad \frac{\partial m^+}{\partial t} \geqslant 0, \qquad \frac{\partial m^-}{\partial t} \leqslant 0$$
$$m_0^+(x,0) = m_0^-(x,0) = 0, \qquad \frac{\partial m_0^+}{\partial t} \geqslant 0, \qquad \frac{\partial m_0^-}{\partial t} \leqslant 0$$

The terms on the right side of Eq. (1.1) represent the reactive forces of the particles being added and detached in their absolute motion.

Equation (1.1) may be written in the form

$$(m + m_0)\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(EF\frac{\partial u}{\partial x}\right) = \frac{\partial m^+}{\partial t}\left(v^+ - \frac{\partial u}{\partial t}\right) + \frac{\partial m^-}{\partial t}\left(v^- - \frac{\partial u}{\partial t}\right) + \frac{\partial m_0^+}{\partial t}\left(v_0^+ - \frac{\partial u}{\partial t}\right) + \frac{\partial m_0^-}{\partial t}\left(v_0^- - \frac{\partial u}{\partial t}\right) \tag{1.3}$$

The right side of Eq. (1.3) represents the reactive forces of the particles in their relative motion as they are added and detached.

When exterior forces act on the rod, it is necessary to introduce terms on the right sides of Eqs. (1.1) and (1.3) which take these forces into account.

If variable discrete masses are associated with the rod at the points $x_1, x_2, \ldots, x_n$, then the force with which the i-th mass acts on the rod is equal to

$$N_i = -\frac{d}{dt}\left(m_i\frac{du}{dt}\right) + \frac{dm_i^+}{dt}v_i^+ + \frac{dm_i^-}{dt}v_i^-$$

Here $u(x_i, t)$ is the displacement of the cross section with the coordinate x_i; $v_i^+(t)$, $v_i^-(t)$ are the speeds of the particles of the i-th mass ($i = 1, 2, \ldots, n$) being added and detached; $m_i^+(t) \geq 0$, $m_i^-(t) \leq 0$ are the masses being added and detached, respectively, to the particle with the coordinate x_i at the time t.

At the initial instant $t = 0$

$$m_i^+(0) = m_i^-(0) = 0, \quad \frac{dm_i^+}{dt} \geqslant 0, \quad \frac{dm_i^-}{dt} \leqslant 0 \quad (i = 1, 2, \ldots, n)$$

The mass of the discretely distributed particles at the time t is equal to

$$\sum_{i=1}^{n} m_i(t) = \sum_{i=1}^{n} (m_i^\circ + m_i^+ + m_i^-)$$

where $m_i^\circ = \text{const}$ is the initial mass of the particle associated with the rod at the point with coordinate x_i.

The equations of the longitudinal oscillations in this case take the form

$$\frac{\partial}{\partial t}\left[(m + m_0)\frac{\partial u}{\partial t}\right] + \sum_{i=1}^{n}\frac{\partial}{\partial t}\left[m_i\frac{\partial u(x_i,t)}{\partial t}\right]\sigma(x - x_i) - \frac{\partial}{\partial x}\left(EF\frac{\partial u}{\partial x}\right) =$$
$$= \frac{\partial m^+}{\partial t}v^+ + \frac{\partial m^-}{dt}v^- + \frac{\partial m_0^+}{\partial t}v_0^+ + \frac{\partial m_0^-}{\partial t}v_0^- + \sum_{i=1}^{n} \times$$
$$\times\left(\frac{dm_i^+}{dt}v_i^+ + \frac{dm_i^-}{dt}v_i^-\right)\sigma(x - x_i) \tag{1.4}$$

$$(m + m_0)\frac{\partial^2 u}{\partial t^2} + \sum_{i=1}^{n} m_i \frac{d^2 u}{dt^2}\,\sigma(x - x_i) - \frac{\partial}{\partial x}\left(EF\frac{\partial u}{\partial x}\right) =$$

$$= \frac{\partial m^+}{\partial t}\left(v^+ - \frac{\partial u}{\partial t}\right) + \frac{\partial m^-}{\partial t}\left(v^- - \frac{\partial u}{\partial t}\right) + \frac{\partial m_0{}^+}{\partial t}\left(v_0{}^+ - \frac{\partial u}{\partial t}\right) +$$

$$+ \frac{\partial m_0{}^-}{\partial t}\left(v_0{}^- - \frac{\partial u}{\partial t}\right) + \sum_{i=1}^{n}\left\{\frac{dm_i{}^+}{dt}\left[v_i{}^+ - \frac{du}{dt}\right] + \right.$$

$$\left. + \frac{dm_i{}^-}{dt}\left[v_i{}^- - \frac{du}{dt}\right]\right\}\sigma(x - x_i) \qquad (1.5)$$

In these equations σ is an impulse function of the first order.

To obtain the equations of torsional oscillations of a rod of variable composition it is necessary in the Eqs. (1.1), (1.3), (1.4), and (1.5) to replace the quantities corresponding to translational motion of a cross section by analogous quantities for rotational motion.

The initial and boundary conditions for these equations are formed in the same way as for the equations of oscillations of rods of constant composition because the reactive forces can be regarded as an external load.

The case should be noted in which particles are added to and detached from a homogeneous rod with zero relative speeds. Here the density and modulus of elasticity of the rod do not vary, and the mass per unit length of the rod is equal to

$$m(x, t) = \rho F_0(x) f(t), \qquad f(t) > 0$$

where ρ is the density, and $F_0(x)$ is the initial area of the cross section.

The equation of the oscillations in this case has the form

$$\frac{\partial^2 u}{\partial t^2} = \frac{a^2}{F_0}\frac{\partial}{\partial x}\left(F_0\frac{\partial u}{\partial x}\right), \qquad \frac{E}{\rho} = a^2 = \text{const}$$

i.e., the oscillations occur as if there were no addition or detachment of particles. This is due to the fact that the areas of the cross sections change in a similar way with the time.

Example. We consider the free oscillations of a rod whose ends are fixed. The density and modulus of elasticity are constant. A separation of particles occurs with the relative speed

$$v^- = (2\alpha + 1)\,\partial u/\partial t \qquad (\alpha > 0)$$

The area of a cross section varies with the time according to the law $F = e^{-t}$. At the initial instant

$$u(x, 0) = f_1(x), \qquad \frac{\partial u(x, 0)}{\partial t} = f_2(x)$$

The equation of the oscillations of the rod takes the form

$$\frac{\partial^2 u}{\partial t^2} + 2\alpha\frac{\partial u}{\partial t} - a^2\frac{\partial^2 u}{\partial x^2} = 0, \qquad a^2 = \frac{E}{\rho} = \text{const}$$

It is identical to the equation of the free oscillations of a rod of constant composition with a resistance proportional to the first power of the speed [2].

2. Equations of Transverse Oscillations. The speeds of the particles being detached and added are perpendicular to the neutral plane. For the magnitude of the velocity of the particles being added (detached) we take the mean value of the speed of the particles being added (detached) on both sides of the neutral plane. To obtain the free deflectional oscillations of a rod of variable composition to which no discrete masses are adjoined we form the functional

$$S_2 = \frac{1}{2}\int_{t_1}^{t_2}\int_0^l \left\{(m + m_0)\left(\frac{\partial y}{\partial t}\right)^2 + J_0\left(\frac{\partial^2 y}{\partial x\,\partial t}\right) - EJ\left(\frac{\partial^2 y}{\partial x^2}\right) - P\left(\frac{\partial y}{\partial x}\right)^2 + \right.$$

$$\left. + 2\left[\frac{\partial m^+}{\partial t}v^+ + \frac{\partial m^-}{\partial t}v^- + \frac{\partial m_0{}^+}{\partial t}v_0{}^+ + \frac{\partial m_0{}^-}{\partial t}v_0{}^-\right]y\right\}dx\,dt$$

where y(x, t) is the transverse displacement of the elastic axis of the rod; $J_0(x, t)$ is the moment of inertia of a unit length of the rod, along with the mutually noninteracting particles fastened to it, relative to the central axis perpendicular to the plane of oscillations; J(x, t) is the moment of inertia of a cross section of the rod relative to the neutral axis of the section perpendicular to the plane of oscillations; P(x, t) is the strength of the longitudinal tensile force.

The equation of the transverse oscillations of a rod of variable composition is the Euler equation for the functional given above, namely,

$$\frac{\partial}{\partial t}\left[(m+m_0)\frac{\partial y}{\partial t}-\right]-\frac{\partial^2}{\partial x\,\partial t}\left(J_0\frac{\partial^2 y}{\partial x\,\partial t}\right)+\frac{\partial^2}{\partial x^2}\left(EJ\frac{\partial^2 y}{\partial t^2}\right)-$$
$$-\frac{\partial}{\partial x}\left(P\frac{\partial y}{\partial x}\right)=\frac{\partial m^+}{\partial t}v^+ +\frac{\partial m^-}{\partial t}v^- +\frac{\partial m_0^+}{\partial t}v_0^+ +\frac{\partial m_0^-}{\partial t}v_0^- \tag{2.1}$$

If the reactive forces in the relative motion of the particles being added and detached are known, then Eq. (2.1) can be written in the form

$$(m+m_0)\frac{\partial^2 y}{\partial t^2}-\frac{\partial^2}{\partial x\partial t}\left(J_0\frac{\partial^2 y}{\partial x\partial t}\right)+\frac{\partial^2}{\partial x^2}\left(EJ\frac{\partial^2 y}{\partial x^2}\right)-\frac{\partial}{\partial x}\left(P\frac{\partial y}{\partial x}\right)=$$
$$=\frac{\partial m^+}{\partial t}\left(v^+-\frac{\partial y}{\partial t}\right)+\frac{\partial m^-}{\partial t}\left(v^--\frac{\partial y}{\partial t}\right)+\frac{\partial m_0^+}{\partial t}\left(v_0^+-\frac{\partial y}{\partial t}\right)+\frac{\partial m_0^-}{\partial t}\left(v_0^--\frac{\partial y}{\partial t}\right) \tag{2.2}$$

If variable masses are fastened to the rod, the equations for such oscillations may be obtained which are analogous to Eqs. (1.4) and (1.5)

$$\frac{\partial}{\partial t}\left[(m+m_0)\frac{\partial y}{\partial t}\right]+\sum_{i=1}^{n}\frac{d}{dt}\left(m_i\frac{dy}{dt}\right)\sigma(x-x_i)-$$
$$-\frac{\partial^2}{\partial x\,\partial t}\left(J_0\frac{\partial^2 y}{\partial x\partial t}\right)+\frac{\partial^2}{\partial x^2}\left(EJ\frac{\partial^2 y}{\partial x^2}\right)-\frac{\partial}{\partial x}\left(P\frac{\partial y}{\partial x}\right)=\frac{\partial m^+}{\partial t}v^+ + \tag{2.3}$$
$$+\frac{\partial m^-}{\partial t}v^- +\frac{\partial m_0^+}{\partial t}v_0^+ +\frac{\partial m_0^-}{\partial t}v_0^- +\sum_{i=1}^{n}\left(\frac{dm_i^+}{dt}v_i^+ +\frac{dm_i^-}{dt}v_i^-\right)\sigma(x-x_i)$$

$$(m+m_0)\frac{\partial^2 y}{\partial t^2}+\sum_{i=1}^{n}m_i\frac{d^2 y}{dt^2}\sigma(x-x_i)-\frac{\partial^2}{\partial x\partial t}\left(J_0\frac{\partial^2 y}{\partial x\partial t}\right)+$$
$$+\frac{\partial^2}{\partial x^2}\left(EJ\frac{\partial^2 y}{\partial x^2}\right)-\frac{\partial}{\partial x}\left(P\frac{\partial y}{\partial x}\right)=\frac{\partial m^+}{\partial t}\left(v^+-\frac{\partial y}{\partial t}\right)+ \tag{2.4}$$
$$+\frac{\partial m^-}{\partial t}\left(v^--\frac{\partial y}{\partial t}\right)+\frac{\partial m_0^+}{\partial t}\left(v_0^+-\frac{\partial y}{\partial t}\right)+\frac{\partial m_0^-}{\partial t}\left(v_0^--\frac{\partial y}{\partial t}\right)+$$
$$+\sum_{i=1}^{n}\left\{\frac{dm_i^+}{dt}\left[v_i^+-\frac{dy}{dt}\right]+\frac{dm_i^-}{dt}\left[v_i^--\frac{dy}{dt}\right]\right\}\sigma(x-x_i)$$

In rods whose length is significantly larger than the transverse dimensions the rotatory inertia can be neglected, and in the Eqs. (2.1)-(2.4) terms containing J_0 can be omitted.

The initial and boundary conditions for these equations are written in the same way as for the equations of transverse oscillations of rods of constant composition.

Example. We consider the free oscillations of a homogeneous rod of rectangular section, hinge-supported at its ends. Material of the same composition as that of the rod is added on to the rod symmetrically with respect to its neutral plane. The mean relative speeds are equal to zero. At the initial instant

$$y(x,0)=f_1(x),\qquad \frac{\partial y(x,0)}{\partial t}=f_2(x)$$

The thickness of a cross section varies according to the law $h=2\sqrt{3e^t}$. Consequently,

$$J/F=e^{2t},\quad E/\rho=a^2=\text{const}$$

We neglect rotatory inertia. The oscillations of such a rod are describable by the equation

$$\frac{\partial^2 y}{\partial t^2}+a^2e^{-2t}\frac{\partial^4 y}{\partial x^4}=0$$

By separating the variables we can obtain the solution of this equation in the form

$$y(x,t)=\sum_{k=1}^{\infty}[A_kI_0(p_ke^t)+B_kY_0(p_ke^t)]\sin\frac{k\pi x}{l}$$
$$p_k=k^2\pi^2al^{-2}$$

$$A_k = \frac{2}{\Delta l} \int_0^l \left[f_2 Y_1 (p_k) + f_2 \frac{Y_0 (p_k)}{p_k} \right] \sin \frac{k\pi x}{l} dx$$

$$B_k = -\frac{2}{\Delta l} \int_0^l \left[f_1 I_1 (p_k) + f_2 \frac{I_1 (p_k)}{p_k} \right] \sin \frac{k\pi x}{l} dx$$

$$\Delta = I_0 \ (p_k) Y_1 (p_k) - I_1 (p_k) Y_0 (p_k) \quad (k=1, 2, \ldots)$$

Here I_0, I_1, Y_0, and Y_1 are Bessel functions of the first and second kinds [3].

We note, in conclusion, that the equations of oscillations of rods of variable composition differ from the equations of oscillations of rods of constant composition in two essential ways: 1) the coefficients for the derivatives can be functions not only of the coordinates but also of the time; 2) the reactive forces are added to the external loads.

It should also be noted that a number of problems which occur in the applied theory of oscillations – parametric oscillations, automatic balancing by addition of masses, automatic damping of the oscillations of beams by application at the nodes of the oscillations (for example, by the addition) of damping coatings, and other examples – may be regarded as special cases of oscillations of rods of variable composition.

LITERATURE CITED

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